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DASC Lab

Self-triggered Control for Safety Critical Systems using Control Barrier Functions

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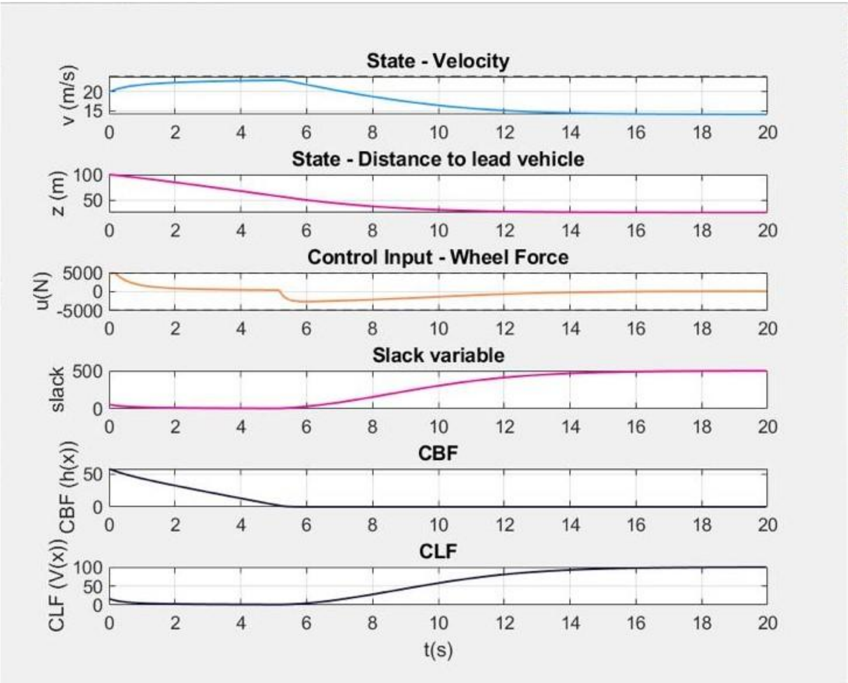
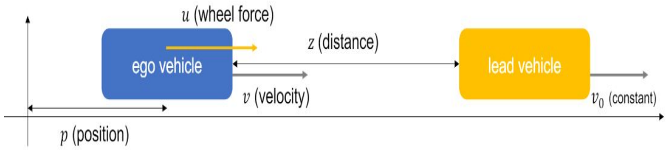
Groundwork

- Started off with understanding the concepts of Control barrier functions and control Lyapunov functions.
- After getting a good grasp on them, worked on implementing a CBF-CLF QP controller for a simple adaptive cruise control system (relative degree 1 system) and on a double integrator system (relative degree 2).
- The adaptive cruise control system example is based on the Control barrier function-based quadratic programs with application to adaptive cruise control paper.

Results from the Adaptive cruise control example.



System dynamics of an adaptive cruise control system:



State

$x = [p \ v \ z]^T \in \mathbb{R}^3$

Control Input

$u \in \mathbb{R}^1$

Dynamics

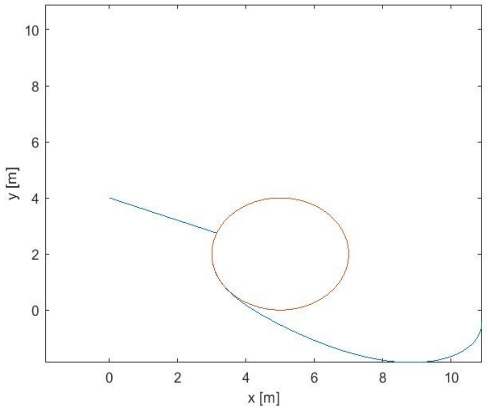
$$\dot{s} = \underbrace{\begin{bmatrix} v \\ -\frac{1}{m}F_r(v) \\ v_0 - v \end{bmatrix}}_{f(s)} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \\ 0 \end{bmatrix}}_{g(s)} u$$

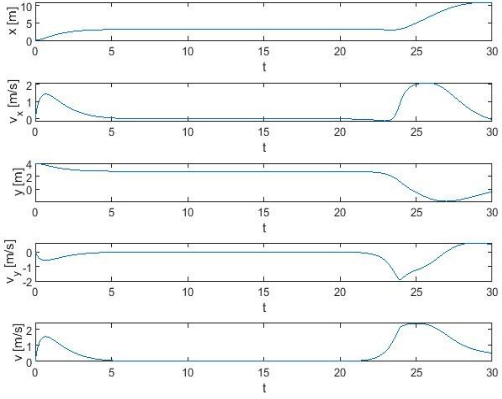
Input constraints $\quad -mc_d g \le u \le mc_a g$

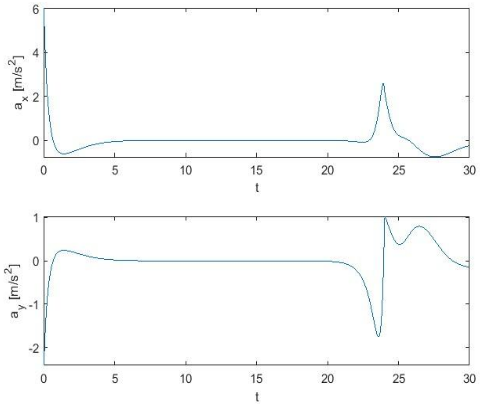
Stability objective $\quad v \to v_d$ (v_d: desired velocity)

Safety objective $\quad z \ge T_h v$ (T_h: lookahead time)

Results from 2D Double integrator example (HOCBF)







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Goal of the project

- Implementation of a real-time control strategy that combines self-triggered control with Control Lyapunov Functions
- Implementation of the controller that overcomes the main limitations of traditional approaches based on periodic controllers, i.e. unnecessary controller updates and potential violations of the safety constraints.
- Central to this approach is the notion of a safe period, which enforces a safety guarantee for implementing ZOH control.

Why self-triggered control for safety critical systems?

- Previous works on CBF-CLF controllers are based on a continuous time formulation, which contradicts the reality that these controllers are implemented on digital platforms.
- Traditionally, digital controllers are implemented using discretized periodic control inputs. A popular discretization method is the Zeroth-Order Hold (ZOH)

  1. Given a fixed update period, there is no guarantee that the safety constraints will hold.
  2. There are unnecessary computations and control updates due to fixed-time sampling.

How do we overcome these issues?

1. To overcome these issues, we can implement self-triggered CBF, The core of all self-triggered controllers consists of two parts. First, a designed feedback controller computes the control input at a given time instance.
2. Second, it determines the next controller update time instance based on sensor measurements and mission requirements.

Introduction

- Implement a self-triggered controller that pre-computes the next update time instance given the current state, control objective, and safety requirements.
- The controller is applied in a ZOH manner.
- To validate the implementation of a self-triggered controller, A second-order double integrator dynamical system is used.
- Compare our self-triggered control strategy with standard periodic control.

Application to second order double integrator system

$$\dot{x} = f(x) + g(x)u,$$

- A continuous time dynamical control affine system.

- The second order double integrator system is defined. And has a relative degree of 2. The goal is to stabilize our system to a desired state.

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Defining the CBF constraints

$$h^{r_b}(x) = \pounds_f^{r_b} h(x) + \pounds_g \pounds_f^{r_b - 1} h(x) u.$$

The time derivative h are related to the Lie derivatives by

$$\xi_b(x) = \begin{bmatrix} h(x) \\ \dot{h}(x) \\ \cdot \\ \cdot \\ h^{r_b}(x) \end{bmatrix},$$

A transverse variable is defined for ECBF.

$$\mathbf{h}(x) = \begin{bmatrix} h_1(x(t)) \\ h_2(x(t)) \\ h_3(x(t)) \\ h_4(x(t)) \end{bmatrix} = \begin{bmatrix} x_1(t) - x_{1,min} \\ -x_1(t) + x_{1,max} \\ x_2(t) - x_{2,min} \\ -x_2(t) + x_{2,max} \end{bmatrix},$$

Given the dynamic system, The ECBF (Exponential control barrier function) is defined as h1(x), h2(x), h3(x), h4(x), as the safety constraints.

$$\inf_{u \in U} [\pounds_f^{r_b} h(x) + \pounds_g \pounds_f^{r_b - 1} h(x) u + K_b \xi_b(x)] \geq 0, \forall x \in Int(C).$$

indicates that the magnitude of the safety constraint is unbounded, and therefore the constraint cannot be violated.

$$\begin{aligned} \zeta_1 &= u + k_1 x_2 + k_2(x_1 - x_{1,min}), \\ \zeta_2 &= -u + k_1(-x_2) + k_2(-x_1 + x_{1,max}), \\ \zeta_3 &= u + k(x_2 - x_{2,min}), \\ \zeta_4 &= -u + k(-x_2 + x_{2,max}). \end{aligned}$$

If $\zeta_i \geq 0$, i = 1, ..., 4 holds, then our system is forward invariant.

Defining the CLF constraints

$$V(x) = \begin{bmatrix} x_1 - x_{1,d} \\ x_2 \end{bmatrix}^T \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} x_1 - x_{1,d} \\ x_2 \end{bmatrix}.$$

The Lyapunov Function candidate for this example is defined as V(X).

$$c_1\|x\|^2 \leq V(x) \leq c_2\|x\|^2,$$
$$\inf_{u \in U}[\mathcal{L}_f V(x) + \mathcal{L}_g V(x)u + \epsilon V(x)] \leq 0, \quad \forall x \in \mathbb{R}^n.$$

Definition: Exponentially stabilizing control Lyapunov function.

- Based on the definition and the Lyapunov candidate our constraint can be defined as:

$$\eta(x) = [2x_2 + (x_1 - x_{1,d})]u + x_2(2(x_1 - x_{1,d}) + x_2) + \epsilon V.$$

CBF-CLF QP formulation.

The QP formulation for system (25) is

$$\begin{aligned}
\min_{u \in U} \quad & u^T u \\
\text{s.t.} \quad & \zeta_i \geq 0, i = 1, \ldots, 4 \\
& \eta \leq 0 \\
& x(t_k) \in Int(C) \\
& u_l \leq u \leq u_u .
\end{aligned}$$

Computation of CBF safe periods

- We find a bound on the system trajectory that exclusively depends on the general properties of the system dynamics.
- Given the dynamical system defined in starting at x(tk) the distance between the trajectory x(t+tk) and x(tk) is bounded by: $\overline{r}_{t_k}(t) = r_0 e^{L(t-t_k)} - \frac{1}{L}\|f(x(t_k)) + g(x(t_k))u_k\|.$
- With lower bound ζ(t), we determine safe period τCBF, such that lower bound ζ (tk+τCBF) = 0.
- The CBF safe period τCBF is then calculated by finding the minimum of the roots of these equations. This was achieved by implementing the Bisection method.

$$
\begin{aligned}
\underline{\zeta_1} &= (k_1(x_2(t_k) - r_{t_k}(t)) - k_2\|u_k\|)t + \zeta_1(t_k),\\
\underline{\zeta_2} &= (-k_1(x_2(t_k) + r_{t_k}(t)) - k_2\|u_k\|)t + \zeta_2(t_k),\\
\underline{\zeta_3} &= -k\|u_k\|t + \zeta_3(t_k),\\
\underline{\zeta_4} &= -k\|u_k\|t + \zeta_4(t_k).
\end{aligned}
$$

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Derivation of the distance bound on system trajectory

$$r_{t_k}(t) = \|x(t_k + t) - x(t_k)\|$$

$$\begin{aligned}\dot{r}(x(t+t_k)) &= \frac{(x(t+t_k) - x(t_k))^T}{\|x(t+t_k) - x(t_k)\|}\dot{x}(t+t_k)\\ &= \frac{(x(t+t_k) - x(t_k))^T}{\|x(t+t_k) - x(t_k)\|}f(x(t+t_k), u).\end{aligned}$$

$$\begin{aligned}\dot{r}_{t_k} &\leq \|f(x(t+t_k), u)\|\\ &\leq \|f(x(t+t_k), u) - f(x(t_k), u) + f(x(t_k), u)\|\\ &\leq \|f(x(t+t_k), u) - f(x(t_k), u)\| + \|f(x(t_k), u)\|,\end{aligned}$$

$$\begin{aligned}\dot{\bar{r}}_{t_k}(t) &= L\|x(t+t_k) - x(t_k)\| + \|f(x(t_k), u)\|\\ &\leq L\bar{r}(t+t_k) + \|f(x(t_k), u)\|.\end{aligned}$$

$$\bar{r}_{t_k}(t) = r_0 e^{L(t-t_k)} - \frac{1}{L}\|f(x(t_k)) + g(x(t_k))u_k\|.$$

$$r_0 = \frac{1}{L}\|f(x(t_k)) + g(x(t_k))u_k\|.$$

Computation of CLF safe period

- To achieve at least asymptotic stability, we must define a CLF update period that guarantees that the Lyapunov function decreases at every step.
- Upper bound of V(t): $\overline{V}(t) \geq V(x(t)), \forall t_{k+1} \geq t \geq t_k,$

$$V(x(t)) \leq V(t_k) + (t - t_k)V'(t_k) + (t - t_k)^2 \frac{D}{2} = \overline{V}(t).$$

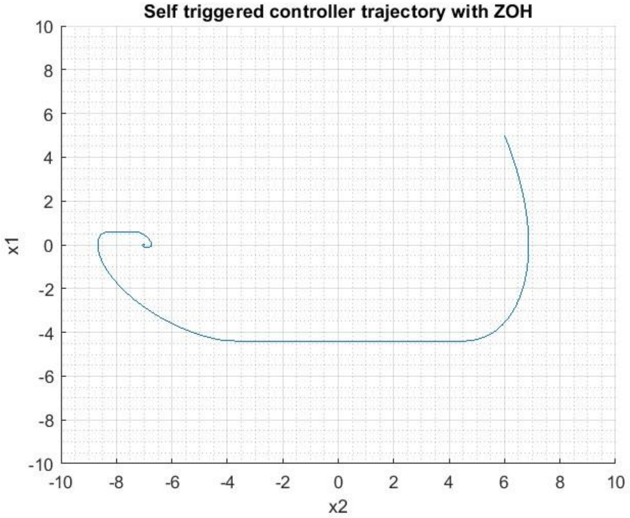
- Since the upper bound of V (t) is a quadratic function in terms of t, there exists a closed-form solution for the roots. The non-zero root is given by: $\tau_{CLF} = \frac{-2V'(x(t_k))}{D}.$

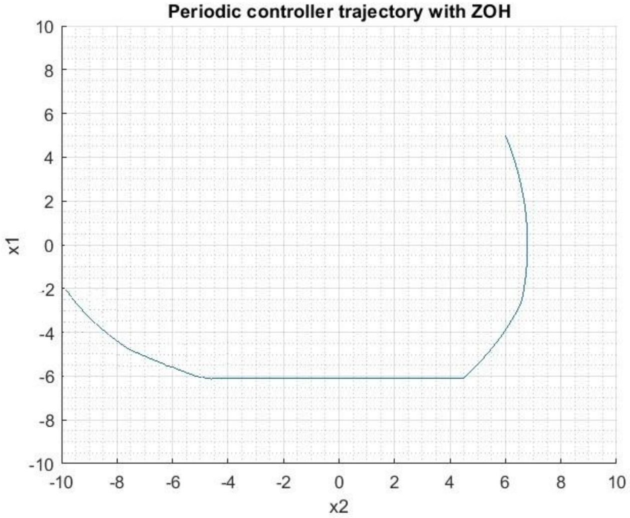
$$V'(x(t_k)) = 2x_2(x_1 - x_{1,d}) + x_2^2 + ((x_1 - x_{1,d}) + 2x_2)u_k$$

- Here the first derivative of V (x(tk)) is given by:
- And the denominator D is given by:

$$D = \max V''(x(t))$$
$$= 2V(x_{t_k}) + 2|u_k|\sqrt{V(x_{t_k})} + 3|\sqrt{V(x_{t_k})}||u_k| + 2|u_k|^2$$

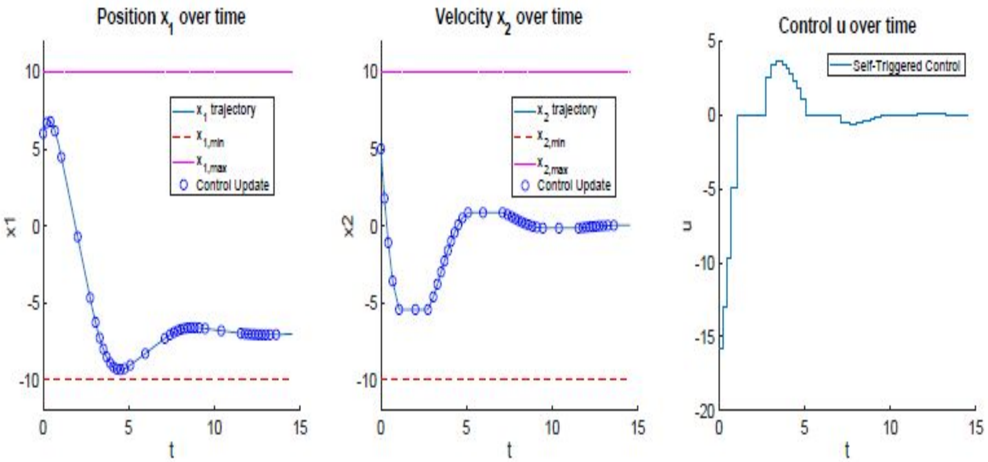
Results: comparing the trajectory generated by both the controllers

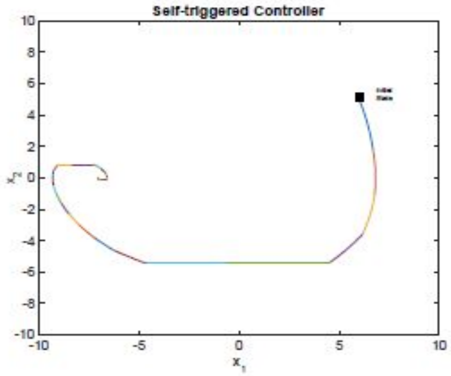




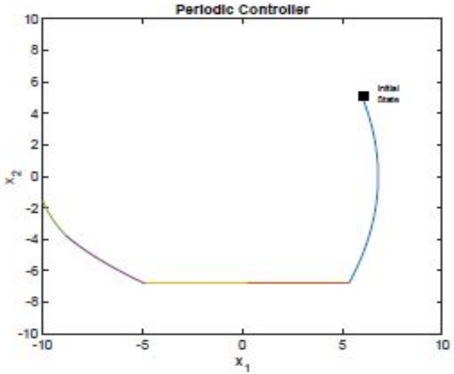
- In the self triggered controller case CBF constraint does not conflict with the input constraints whereas in the periodic controller case it does, the position x1 violates x1,min.

Comparing the results from the paper.



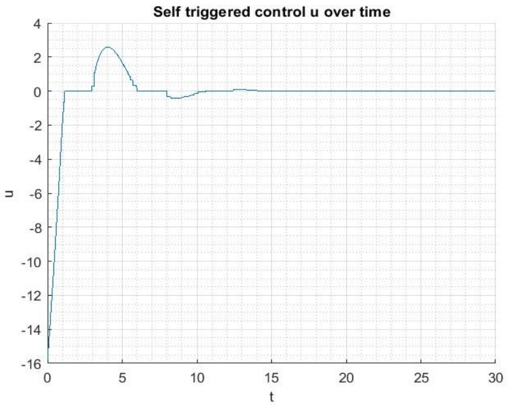


(a)

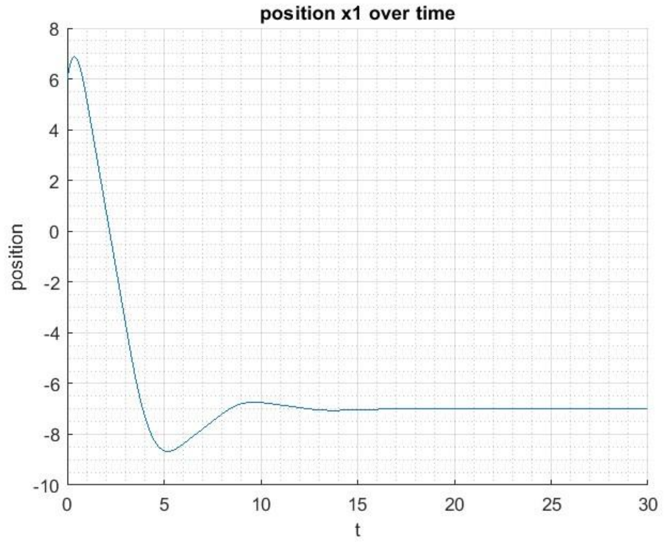


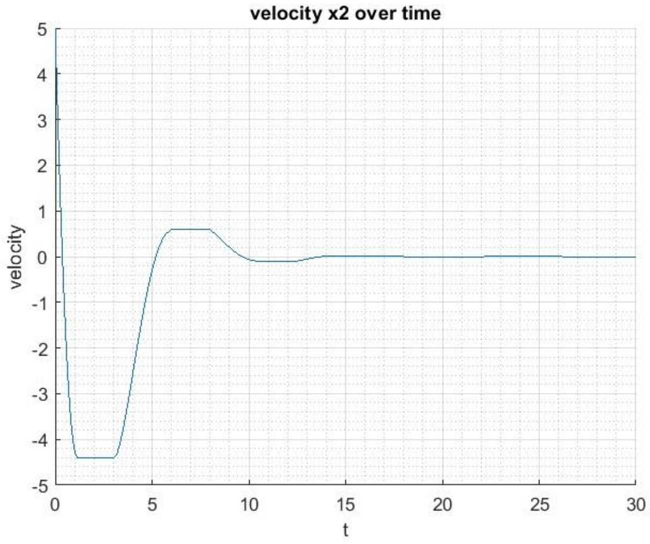
Results: Optimal control generated by self-triggered controller

- For self-triggered control, the controller only updates when the system is about to violate CBF constraints, or the system is deviating away from the desired states.
- the update interval for self-triggered controller becomes a lot faster as the system approaches to the unsafe region (x1 < x1,min) in order to prevent violation on safety constraint.



Result: Position and velocity of the system for self-triggered controller





Conclusion and remarks

- The self-triggered controller is designed to obtain the safety property from CBFs while stabilizing a system asymptotically.
- In this example, since we cannot directly control the state x1, the use of ECBF becomes necessary. ECBF framework is used to ensure the controller works for system with high relative degrees.
- The CLF safe period follows an increasing trend as the system reaches the equilibrium state which contradicts the results of the paper. (working on resolving this issue)